EQUATIONS OF TURBULENT MOTION

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Turbulence occurs in numerous types of liquid and gas flows both in nature and in man-made apparatus. It consists in the presence of chaotic pulsations (i.e. very irregular variations in space and time) of the velocity, pressure, temperature, and other hydrodynamic characteristics of these flows.

We note to begin with that the scales of space inhomogeneities of the hydrodynamic fields of turbulent flow cannot be too small, since very small inhomogeneities would be associated with very large velocity gradients. Such motions are practically impossible because of the very large amounts of energy required to overcome the forces of viscous friction. Hence, the minimal spatial scales λ and periods τ of turbulent pulsations (according to Kolmogorov [1] these are given by Formulas $\lambda \sim (\nu^3/\varepsilon)^{1/4}$ and $\tau \sim (\nu/\varepsilon)^{1/2}$, where ν is the kinematic coefficient of molecular viscosity and ε is the rate of viscous dissipation of kinetic energy per unit mass) are, under ordinary conditions, several orders larger than the scales and periods of molecular motions. In air at normal pressure, for example, $\lambda \approx 0.1$ cm, and the free path of the molecules is on the order of 10^{-5} cm; moreover, since turbulent velocity pulsations do not exceed the average velocity of thermal motion of the molecules (close to 5×10^4 cm/sec) in order of magnitude, the values of $\tau (\approx 0.1$ sec) exceed the average time between molecular collisions (10^{-9} sec) by many orders.

At distances comparable with λ and over time intervals comparable with τ all hydrodynamic fields vary smoothly and can be described by means of differentiable functions. Hence, turbulent flows are fully describable by means of the ordinary differential equations of hydromechanics (e.g. the Navier-Stokes equations). It is therefore unnecessary to return (as is sometimes suggested) to the equations of the kinetic theory of gases in order to describe turbulence. (The latter statement is true except in special cases such as that of highly rarefied gases in which the internal turbulence scale λ is comparable or even exceeds the mean free path of the molecules. This would apply, for example, in the upper atmosphere, where λ attains values of several tens of meters at altitudes above 100 km, hundreds of meters above 120 km, and thousands of meters above 140 km.)

Furthermore, reversion to the equations of kinetic gas theory only adds to the difficulties involved in the closure problem of turbulence theory (which we shall consider below) and introduces additional difficulties arising from the necessity of converting from the concepts and equations of kinetic gas theory to the concepts and equations of macroscopic hydromechanics (this conversion is, in fact, not so very simple: recall the derivation of hydromechanics equations from the equations of kinetic gas theory by the Enskog-Chapman method). Without going deeper into this question, we shall proceed from the fact that turbulent motions can be described by means of the ordinary differential equations of hydromechanics.

We must bear in mind, however, that the use of hydromechanics equations for the exact description of all the details of an individual turbulent flow is only a theoretical possibility. Such description is practically impossible due to the extreme irregularity of the hydrodynamic fields of turbulent flows. If we characterize the flow of a liquid or gas as a nonlinear mechanical system with a very large number of degrees of freedom (or generalized coordinates which, for example, can take the form of the coefficients of expansion of the velocity field in some complete system of functions of the space coordinates), it turns out that an enormous number N of degrees of freedom is excited at all times (according to Landau and Lifshits [2] in the case of flow in a bounded volume $N \approx (N_{Re}/N_{Re^*})^{9/4}$, where Re is the Reynolds number and Re* is its critical value). This means that the variations in space and time of every hydrodynamic characteristic are described by functions containing an enormous number N of Fourier components. Moreover, hydrodynamic turbulent flow fields depend strongly on the most minute details of the initial conditions. Since these are never known in sufficient detail, it follows that exact solutions of the hydromechanics equations would be extremely cumbersome and practically useless because of their instability with respect to small perturbations of the initial data. In fact, the only expedient and feasible approach is a statistical description of turbulent flows based on the investigation of the statistical properties of the ensemble of turbulent flows under macroscopically equivalent external conditions.

1. Functional formulation of the turbulence problem. Let us formulate the problem of complete statistical description of turbulent flows (sometimes called the "turbulence problem") in mathematical language, limiting ourselves for simplicity to the case of an incompressible fluid whose flows are completely characterized by their solenoidal (i.e. nondivergent) velocity fields $\mathbf{u}(\mathbf{x}, t)$; the pressure p can be expressed in terms of the velocity field at the same instant by means of Formula

$$p(\mathbf{x}, t) = -\rho\Delta^{-1}(\mathbf{x}, \mathbf{x}') \frac{\partial^2 u_\alpha(\mathbf{x}', t)\, u_\beta(\mathbf{x}', t)}{\partial x_\alpha' \partial x_\beta'} = \frac{\rho}{4\pi}\int \frac{\partial^2 u_\alpha(\mathbf{x}', t)\, u_\beta(\mathbf{x}', t)}{\partial x_\alpha' \partial x_\beta'} \frac{\partial x'}{|\mathbf{x}-\mathbf{x}'|} \quad (1.1)$$

where ρ is the constant density of the fluid; Δ^{-1} is an integral operator which is the inverse of the Laplacian (the recurrent Greek-letter subscripts denote summation). Let $\Omega = (\omega)$ be the phase space of the turbulent flow of the incompressible fluid, i.e. the set whose points ω are all the possible solenoidal vector fields $\mathbf{u}(\mathbf{x}, t)$ which satisfy the hydromechanics equations and the appropriate boundary conditions at the stream boundaries; we assume that some topology is specified on this set, so that Ω is a linear topological functional space. The turbulence problem then consists in finding the probability distribution in the phase space, i.e. in determining the probability measure $P(d\Omega)$ on Ω. Assumption of the existence of such a probability distribution is equivalent to interpreting the velocity field $\mathbf{u}(\mathbf{x}, t)$ of the turbulent flow as a random field.

From the purely mathematical standpoint the phase space Ω is finite-dimensional. Determination on an infinite-dimensional space of a measure $P(d\Omega)$ with properties (*) rendering it convenient for analysis is not a simple matter (e.g. see the monograph by Gel'fand and Vilenkin [3]). Furthermore, since a volume element cannot be defined in an infinite-dimensional space, the probability distribution $P(d\Omega)$ does not have a probability density. Hence, in order to avoid operations involving functions of the sets $P(S)$ and $S \subset \Omega$, which are rather difficult to analyze, we can investigate not the probability distribution $P(d\Omega)$, but rather its "Fourier transform", i.e. the characteristic functional

$$\Phi[\theta(\mathbf{x}, t)] = \langle \exp\{i(\theta \cdot \mathbf{u})\}\rangle = \int e^{i(\theta \cdot \mathbf{u})} P(d\Omega) \quad (1.2)$$

where $(\theta \cdot \mathbf{u})$ represents the integral over $d\mathbf{x}dt$ of the scalar product of the random function $\mathbf{u}(\mathbf{x}, t)$ and the nonrandom function $\theta(\mathbf{x}, t)$ taken over the entire volume occupied by the fluid; the square brackets here and below denote the mathematical expectation of this expression (i.e. its integral over the measure $P(d\Omega)$). The notion of the characteristic functional was first introduced in 1935 by Kolmogorov [4] (for probability distributions in Banach spa-

*) A particularly desriable property is that of countable additivity, which would, for example, guarantee the interchangeability of the operation of integration over this measure of functions in ω, i.e. of the mathematical expectation operator of functionals in $\mathbf{u}(\mathbf{x}, t)$, with limiting processes including the differentiation and integration of these functionals over the parameter

ces). The necessary and sufficient conditions for the functional $\Phi[\theta(\mathbf{x}, t)]$ to be the characteristic functional of some countably additive probability measure (among these conditions is that the function be nonnegatively defined, that it equal unity at zero, and that it be continuous in some topology), and also the conditions for the unique definition of the measure by its characteristic functional were investigated by Prohorov [5]. We note, however, that the above mathematical difficulties arising from the finite dimensionality of the phase space Ω are devoid of physical significance, so that the number N of degrees of freedom of the turbulent flow, and therefore the number N of dimensions of the phase space, is very large, but nevertheless finite (as noted above).

Thus, we can assume that determination of the characteristic functional is equivalent to the solution of the turbulence problem. In determining the characteristic functional we can make use of the fact that it must satisfy certain dynamic equations which follow from the hydromechanics equations. In order to formulate these equations we introduce the notion of the variational derivative of the functional $\Phi[\theta(\mathbf{x}, t)]$ with respect to the functional argument $\theta_j(\mathbf{x}, t)$, setting

$$D_i(\mathrm{x}', t')\,\Phi[\theta(\mathrm{x}, t)] = \lim_{\substack{|\delta_j\theta(\mathrm{x}, t)|\to 0\\ V\to 0}} \frac{\Phi[\theta(\mathrm{x}, t) + \delta_j\theta(\mathrm{x}, t)] - \Phi[\theta(\mathrm{x}, t)]}{\int_V \delta_j\theta(\mathrm{x}, t)\,d\mathrm{x}\,dt} =$$

$$= \lim_{h\to 0}\frac{\partial}{\partial h}\Phi[\theta(\mathrm{x}, t) + h\mathrm{e}_j\delta(\mathrm{x}-\mathrm{x}')\,\delta(t-t')] \qquad (1.3)$$

where $\delta_j\theta(\mathbf{x}, t)$ is a vector function with only its j-th component not equal to zero (even this component is nonzero only in a small neighborhood V of the point $(\mathbf{x}', t')$; $\mathbf{e}_j$ is the unit vector along the x_j-axis. In the case of characteristic functional (1.2) variational derivative (1.3) is given by

$$D_j(\mathrm{x}', t')\,\Phi[\theta(\mathrm{x}, t)] = i\,\langle u_j(\mathrm{x}', t')\exp\{i\,(\theta\cdot u)\}\rangle \qquad (1.4)$$

Recalling that the quantity $\exp\{i(\theta\cdot\mathbf{u})\}$ is constant in space and time, differentiating both sides of Eq. (1.4) with respect to x_j', summing over j, and making use of the equation of discontinuity $\partial u_\alpha(\mathbf{x}', t')/\partial x_\alpha' = 0$, we obtain the following equation in first-order variational derivatives for functional (1.2):

$$\frac{\partial}{\partial x_\alpha}\{D_\alpha(\mathbf{x}, t)\,\Phi\} = 0 \qquad (1.5)$$

This equation is equivalent to the condition of solenoidality of the velocity field (it is also equivalent to the requirement that the values of the functional $\Phi[\theta(\mathbf{x}, t)]$ remain unchanged upon the addition to its functional argument $\theta(\mathbf{x}, t)$ of the arbitrary potential term $\nabla\varphi(\mathbf{x}, t)$, or to the requirement that the functional Φ depend only on the solenoidal component $\theta^\circ(\mathbf{x}, t)$ of its argument). Furthermore, differentiating both sides of Eq. (1.4) with respect to t', expressing $\partial u_j/\partial t'$ in terms of the space derivatives of the velocity field by means of the Navier-Stokes equations, and once again applying Eq. (1.4), we obtain the following dynamic equation for the characteristic functional Φ:

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right) D_j\Phi = i\left(\delta_{j\beta} - \frac{\partial}{\partial x_j}\Delta^{-1}\frac{\partial}{\partial x_\beta}\right)\frac{\partial D_\alpha D_\beta\Phi}{\partial x_\alpha} \qquad (1.6)$$

We must solve this equation for a statistically defined initial velocity field $\mathbf{u}_0(\mathbf{x}) = \mathbf{u}(\mathbf{x}, 0)$, i.e. under the condition that on the functions $\theta(\mathbf{x}, t) = \theta(\mathbf{x})\,\delta(t)$ the functional $\Phi[\theta(\mathbf{x}, t)]$ become the given characteristic functional $\Phi_0[\theta(\mathbf{x})]$ of the initial velocity field. A dynamic Eq. of the (1.6) type for the characteristic functional of the velocity field of a turbulent flow of an incompressible fluid was first obtained by Hopf [6] (although for a less complete statistical characteristic of the velocity field, i.e. for its space characteristic functional, which we shall soon discuss). Eq. (1.6) is the most complete and compact form of the equations of turbulent motion.

A remarkable property of Eq. (1.6) is the fact that it is linear. Thus, although fluid dynamics is nonlinear (the evolution of an individual velocity field $\mathbf{u}(\mathbf{x}, t)$ is described by nonlinear equations), the basic problem of the statistical dynamics of turbulent flows (the turbulence problem) is a linear one. The characteristic functional Φ is therefore subject to the superposition principle: if the initial functional Φ_0 is a linear combination of functionals $\Phi_0^{(\gamma)}$, then Φ is a similar linear combination of functionals $\Phi^{(\gamma)}$ which are solutions of Eq. (1.6) for the given initial $\Phi_0^{(\gamma)}$.

A formulation of the turbulence problem which is somewhat more general than (1.6) results from describing turbulent fluid flows in a field of prescribed random forces $\mathbf{X}(\mathbf{x}, t)$ by means of the compatible characteristic functional of the fields $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{X}(\mathbf{x}, t)$ defined by Formula

$$\Phi\,[\theta(\mathbf{x}, t), \mathbf{f}(\mathbf{x}, t)] = \langle \exp\{i(\theta\cdot\mathbf{u}) + i(\mathbf{f}\cdot\mathbf{X})\}\rangle \tag{1.7}$$

whose dynamic equation can be obtained by adding to the right side of (1.6) the term

$$\left(\delta_{j\beta} - \frac{\partial}{\partial x_j}\Delta^{-1}\frac{\partial}{\partial x_\beta}\right) D_{f_\beta}\Phi$$

where $D_{f\beta}$ is the variational derivative with respect to $f_\beta(\mathbf{x}, t)$. Specific problems for such an equation are formulated in [7], for example. On the other hand, a formulation of the turbulence problem which is narrower than (1.6) but is nevertheless adequate for many purposes involves description of the turbulent velocity field $\mathbf{u}(\mathbf{x}, t)$ at a fixed instant t by means of the characteristic functional $\Phi[\theta(\mathbf{x}); t]$ given by the same Formula (1.2) in which $(\theta\cdot\mathbf{u})$ now represents the integral of the scalar product $\theta(\mathbf{x})\cdot\mathbf{u}(\mathbf{x}, t)$ with respect to $d\mathbf{x}$ only; such a functional can be called a space functional, while $\Phi[\theta(\mathbf{x}, t)]$ is a space-time functional. The dynamic equation for the space characteristic functional of the velocity field is given by

$$\frac{\partial\Phi}{\partial t} = \left(\theta^\circ\cdot\left\{i\frac{\partial}{\partial x_\alpha}D_\alpha + \nu\Delta\right\}\mathbf{D}\Phi\right) \tag{1.8}$$

where θ° is the solenoidal component of the vector field $\theta(\mathbf{x})$ which (by virtue of (1.5)) is the only variable on which the functional Φ depends; $\mathbf{D}$ is a vector operator with the components $D_j(\mathbf{x})$. The solution of Eq. (1.8) under the given initial condition $\Phi[\theta(\mathbf{x}); 0] = \Phi_0[\theta(\mathbf{x})]$ yields a complete statistical description of the velocity field $\mathbf{u}(\mathbf{x}, t)$ at each fixed instant t.

Equations of turbulent motion (1.6) or (1.8) can sometimes be expressed conveniently in a spectral form in which the argument of the characteristic functional is not the function θ, but rather its Fourier transform ([8], Sec. 28).

2. Equations for finite-dimensional probability distributions. Let $d\Omega$ be a cylindrical set of elements of the phase space Ω of turbulent flow consisting of all those functions $\mathbf{u}(\mathbf{x}, t)$ which satisfy the hydromechanics equations and the boundary conditions which at fixed points n of space-time $M_m = (\mathbf{x}_m, t_m)$, $(m = 1,\ldots, n)$ assume the values $\mathbf{u}(M_m)$ satisfying the conditions $u_{mj} < u_j(M_m) \leqslant u_{mj} + du_{mj}$ $(j = 1, 2, 3)$. The probability measure $P(d\Omega)$ of such a cylindrical set can be written as

$$P(d\Omega) = p_{M_1,\ldots,M_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n)\,d\mathbf{u}_1,\ldots,d\mathbf{u}_n \tag{2.1}$$

where $p_{M_1,\ldots,M_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n)$ is the $3n$-dimensional probability density of the random quantities $\mathbf{u}_1 = \mathbf{u}(M_1),\ldots, \mathbf{u}_n = \mathbf{u}(M_n)$. Under certain general conditions the knowledge of all the probability densities $p_{M_1,\ldots,M_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n)$ for values of the field $\mathbf{u}(\mathbf{x}, t)$ on all possible finite sets of points of space-time permits complete construction of the measure $P(d\Omega)$. In other words, knowledge of all these densities can constitute a complete statistical description of the random field $\mathbf{u}(\mathbf{x}, t)$. Thus, the problem of turbulence reduces to the determination of all of the finite-dimensional probability distributions $p_{M_1,\ldots,M_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n)$. Conversely, if characteristic functional (1.2) is known, then the finite-dimensional probability distributions can be found from its values on linear combinations of delta functions, since these values are given by

$$\Phi\left[\sum_{m=1}^{n}\theta_m\delta(\mathbf{x}-\mathbf{x}_m)\,\delta(t-t_m)\right]=\left\langle\exp\left\{i\sum_{m=1}^{n}\theta_m\cdot\mathbf{u}(\mathbf{x}_m,\ t_m)\right\}\right\rangle=\varphi_{M_1,\ldots,M_n}(\theta_1,\ldots,\theta_n) \tag{2.2}$$

i.e., since they are characteristic functions of finite-dimensional probability distributions; the latter can be found from Formulas

$$p_{M_1,\ldots,M_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n)=$$
$$=\frac{1}{(2\pi)^{3n}}\int\exp\left[-i\sum_{m=1}^{n}\theta_m\cdot\mathbf{u}_m\right]\varphi_{M_1,\ldots,M_n}(\theta_1,\ldots,\theta_n)\,d\theta_1,\ldots,d\theta_n \tag{2.3}$$

The characteristic functions for the values of u at the points $M_1,\ldots,M_n$ for the same t can also be obtained from the space functional $\Phi[\theta(\mathbf{x});\ t]$,

$$\Phi\left[\sum_{m=1}^{n}\theta_m\delta(\mathbf{x}-\mathbf{x}_m);\ t\right]=\varphi_{\mathbf{x}_1,\ldots,\mathbf{x}_n}(\theta_1,\ldots,\theta_n;\ t) \tag{2.4}$$

The condition of solenoidality of the velocity field in terms of the characteristic functions of the finite-dimensional probability distributions can be written as

$$[\nabla_{\mathbf{x}}\cdot\nabla_\theta\varphi_{\mathbf{x}}(\theta,\ t)]_{\theta=0}=0 \tag{2.5}$$

regardless of whether the function φ depends on any other arguments $M_1,\ldots,M_n$ and $\theta_1,\ldots,\theta_n$.

Closed dynamic equations for characteristic functions (2.2) or (2.4) are unobtainable. In fact, even if we neglect completely all the nonlinear terms of the Navier-Stokes equations (i.e. if we take the latter simply in the form $\partial\mathbf{u}/\partial t=\nu\Delta\mathbf{u}$) the space characteristic functional of the velocity field is given by

$$\Phi[\theta(\mathbf{x});\ t]=\Phi_0\left[\int\exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{4\nu t}\right)\frac{\theta(\mathbf{x}')}{(4\pi\nu t)^{3/2}}d\mathbf{x}'\right] \tag{2.6}$$

while for $\theta(\mathbf{x})=\sum_{m=1}^{n}\theta_m\delta(\mathbf{x}-\mathbf{x}_m)$ we have

$$\varphi_{\mathbf{x}_1,\ldots,\mathbf{x}_n}(\theta_1,\ldots,\theta_n;\ t)=\Phi_0\left[\sum_{m=1}^{n}\frac{\theta_m}{(4\pi\nu t)^{3/2}}\exp\left(\frac{|\mathbf{x}-\mathbf{x}_m|^2}{4\nu t}\right)\right]=$$
$$=\left\langle\exp\left\{i\sum_{m=1}^{n}\left[\theta_m\cdot\int\exp\left(-\frac{|\mathbf{x}-\mathbf{x}_m|^2}{4\nu t}\right)\frac{\mathbf{u}_0(\mathbf{x})\,d\mathbf{x}}{(4\pi\nu t)^{3/2}}\right]\right\}\right\rangle \tag{2.7}$$

i.e. this quantity depends on the values of the initial velocity field $\mathbf{u}_0(\mathbf{x})$ not only at the points $\mathbf{x}_1,\ldots,\mathbf{x}_n$, but also in all continuous space, so that it cannot be expressed solely in terms of the values $\varphi_{\mathbf{x}_1,\ldots,\mathbf{x}_n}(\theta_1,\ldots,\theta_n;0)$. We can only obtain equations expressing the derivatives of the n-point characteristic functions with respect to time in terms of the values of these functions themselves and of the $(n+1)$- point characteristic functions. For functions (2.2) with distinct $t_1,\ldots,t_n$ and distinct $x_1,\ldots,x_n$ these equations are of the form

$$\left[\frac{\partial}{\partial t_m}-i(\nabla_{\mathbf{x}_m}\cdot\nabla_{\theta_m})\right]\varphi_{M_1,\ldots,M_n}(\theta_1,\ldots,\theta_n)= \tag{2.8}$$
$$=-i\,[(\theta_m\cdot\nabla_{\mathbf{x}'})\,\Delta^{-1}(\mathbf{x}',\ \mathbf{x})(\nabla_{\mathbf{x}}\cdot\nabla_\theta)^2\varphi_{M_1,\ldots,M_nM}(\theta_1,\ldots,\theta_n,\theta)]_{\theta=0,\ \mathbf{x}'=\mathbf{x}_m}+$$
$$+\nu\,[\Delta_{\mathbf{x}}(\theta_m\cdot\nabla_\theta)\,\varphi_{M_1,\ldots,M_nM}(\theta_1,\ldots,\theta_n,\theta)]_{\theta=0,\ \mathbf{x}=\mathbf{x}_m}\qquad(M=(x,\ t_m))$$

Making use of (2.8), we obtain the following dynamic equations for probability density (2.3):

$$\left[\frac{\partial}{\partial t_m}+(\mathbf{u}_m\cdot\nabla_{\mathbf{x}_m})\right]p_{M_1,\ldots,M_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n)=$$

$$= -\int [(\nabla_{\mathbf{u}_m}\cdot\nabla_{\mathbf{x}'})\,\triangle^{-1}(\mathbf{x}',\mathbf{x})\,(\mathbf{u}\cdot\nabla_{\mathbf{x}})^2\, p_{M_1,\ldots,M_nM}(\mathbf{u}_1,\ldots,\mathbf{u}_n,\mathbf{u})]_{\mathbf{x}'=\mathbf{x}_m}d\mathbf{u} - \nu\int(\mathbf{u}\cdot\nabla_{\mathbf{u}_m})\,[\triangle_{\mathbf{x}}p_{M_1,\ldots,M_nM}(\mathbf{u}_1,\ldots,\mathbf{u}_n,\mathbf{u})]_{\mathbf{x}=\mathbf{x}_m}d\mathbf{u} \tag{2.9}$$

The dynamic equations for functions (2.4), which result from (2.2) when we set $t_1 = \ldots = t_n = t$, can be obtained by summing Eqs. (2.8) over all m with allowance for the validity in this case of the Eq $\Sigma\partial\varphi/\partial t_m = \partial\varphi/\partial t$. Similarly, from (2.9) we can obtain an equation for the probability density $P_{\mathbf{x}_1,\ldots,\mathbf{x}_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n;\,t)$. Eqs. (2.8) and (2.9) are somewhat simpler when expressed in spectral form. Specifically, let us assume that we can give meaning to the spectral representations

$$\varphi_{M_1,\ldots,M_n}(\theta_1,\ldots,\theta_n) = \int \exp\left[i\sum_{m=1}^{n}\mathbf{k}_m\cdot\mathbf{x}_m\right]\psi_{Q_1,\ldots,Q_n}(\theta_1,\ldots,\theta_n)\,d\mathbf{k}_1,\ldots,d\mathbf{k}_n \tag{2.10}$$

$$p_{M_1,\ldots,M_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n) = \int \exp\left[i\sum_{m=1}^{n}\mathbf{k}_m\cdot\mathbf{x}_m\right]\pi_{Q_1,\ldots,Q_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n)\,d\mathbf{k}_1,\ldots,d\mathbf{k}_n$$

where $Q_m = (\mathbf{k}_m,\,t_m)$. Here the generalized functions $\pi_{Q_1,\ldots,Q_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n)$ and $\psi_{Q_1,\ldots,Q_n}(\theta_1,\ldots,\theta_n)$ are related by the same Fourier transform (2.3) as the functions $P_{M_1,\ldots,M_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n)$ and $\varphi_{M_1,\ldots,M_n}(\theta_1,\ldots,\theta_n)$. We then obtain from (2.8) the following equations for the functions $\psi_{Q_1,\ldots,Q_n}(\theta_1,\ldots,\theta_n)$:

$$\left[\frac{\partial}{\partial t_m} + (\mathbf{k}_m\cdot\nabla_{\theta_m})\right]\psi_{Q_1,\ldots,Q_n}(\theta_1,\ldots,\theta_n) = \int d\mathbf{k}\left\{\left[\frac{(\mathbf{k}\cdot\theta_m)\,(\mathbf{k}\cdot\nabla_\theta)^2}{k^2} - \nu k^2(\theta_m\cdot\nabla_\theta)\right]\psi_{Q_1,\ldots,Q_{m-1}Q_m'Q_{m+1}\ldots Q_nQ}(\theta_1,\ldots,\theta_n,\theta)\right\}_{\theta=0} \tag{2.11}$$

$$(Q_m' = (\mathbf{k}_m - \mathbf{k},\,t_m),\; Q = (\mathbf{k},\,t_m))$$

Similarly, (2.9) yields the following equations for the functions $\pi_{Q_1,\ldots,Q_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n)$:

$$\left[\frac{\partial}{\partial t_m} + i(\mathbf{k}_m\cdot\mathbf{u}_m)\right]\pi_{Q_1,\ldots,Q_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n) = \int d\mathbf{k}d\mathbf{u}\left[\nu k^2(\mathbf{u}\cdot\nabla_{\mathbf{u}_m}) - i\,\frac{(\mathbf{k}\cdot\mathbf{u})^2\,(\mathbf{k}\cdot\nabla_{\mathbf{u}_m})}{k^2}\right]\pi_{Q_1,\ldots,Q_{m-1}Q_m'Q_{m+1},\ldots,Q_nQ}(\mathbf{u}_1,\ldots,\mathbf{u}_n,\mathbf{u}) \tag{2.12}$$

Specifically, let us write out the equation for the one-point probability distribution of the velocity field, e.g. in the form (2.9). After an obvious change in notation, we can write this equation in the form

$$\left[\frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\right]p(\mathbf{u}\,|\,\mathbf{x},\,t) = -\int[(\nabla_{\mathbf{u}}\cdot\nabla_{\mathbf{x}'})\,\triangle^{-1}(\mathbf{x}',\mathbf{x}_1)\,(\mathbf{u}_1\cdot\nabla_{\mathbf{x}_1})^2 p(\mathbf{u}\,|\,\mathbf{x},\,t;\,\mathbf{u}_1\,|\,\mathbf{x}_1,\,t)]_{\mathbf{x}'=\mathbf{x}}d\mathbf{u}_1 - \nu\int(\mathbf{u}_1\cdot\nabla_{\mathbf{u}})\,[\triangle_{\mathbf{x}_1}p(\mathbf{u}\,|\,\mathbf{x},\,t;\,\mathbf{u}_1\,|\,\mathbf{x}_1,\,t)]_{\mathbf{x}_1=\mathbf{x}}d\mathbf{u}_1 \tag{2.13}$$

Any of the equivalent systems of Eqs. (2.8), (2.9), (2.11), or (2.12) can be regarded as a new form of the equations of turbulent motion (such equations have not, to our knowledge, appeared in print previously) (*). Analytically, these equations are, of course, simpler than the equations for characteristic functionals (1.6) or (1.8), since the latter contain variational derivatives instead of the partial derivatives ∇_{θ_m} or $\nabla_{\mathbf{u}_m}$. On the other hand, the new

*) **Remark (during proofreading).** After submission of the present paper for publication there appeared Lundgren's study [24] in which equations for one-point and two-point probability distribution functions are obtained in forms similar, for example, to (2.9) and (2.13).

equations are not closed; the number of unknown functions is always larger than the number of equations in this case, so that their use involves the problem of closure.

3. Equations for moments. The simplest statistical characteristics of a turbulent velocity field are its moments as given by any one of Formulas

$$B_{j_1,\ldots,j_n}(M_1,\ldots,M_n) = \langle u_{j_1}(M_1)\ldots u_{j_n}(M_n)\rangle = \tag{3.1}$$

$$= \int u_{1j_1}\ldots u_{nj_n} p_{M_1,\ldots,M_n}(\mathbf{u}_1,\ldots,\mathbf{u}_n)\,d\mathbf{u}_1\ldots d\mathbf{u}_n =$$

$$= (-i)^n \left[\frac{\partial^n \varphi_{M_1,\ldots,M_n}(\theta_1,\ldots,\theta_n)}{\partial\theta_{1j_1}\ldots\partial\theta_{nj_n}}\right]_{\theta_1=\ldots=\theta_n=0} =$$

$$= (-i)^n \{D_{j_1}(M_1)\ldots D_{j_n}(M_n)\,\Phi[\theta(M)]\}_{\theta(M)\equiv 0}$$

where some of the subscripts $j_1,\ldots, j_n$ and some of the points $M_1,\ldots, M_n$ may coincide. Under certain general conditions knowledge of all the moments enables us to reconstruct finite-dimensional probability distributions by making use of Taylor series for their characteristic functions

$$\varphi_{M_1,\ldots,M_n}(\theta_1,\ldots,\theta_n) = 1 + \sum_{N=1}^{\infty}\frac{i^N}{N!}\sum_{m_1,\ldots,m_N=1}^{n} B_{\alpha_1,\ldots,\alpha_N}(M_{m_1},\ldots,M_{m_N})\theta_{m_1\alpha_1}\ldots\theta_{m_N\alpha_N} \tag{3.2}$$

Similarly, under certain general conditions knowledge of all the moments of the velocity field makes it possible to reconstruct its characteristic functional with the aid of the functional Taylor series

$$\Phi[\theta(M)] = 1 + \sum_{N=1}^{\infty}\frac{i^N}{N!}\int B_{\alpha_1,\ldots,\alpha_N}(M_1,\ldots,M_N)\,\theta_{\alpha_1}(M_1)\ldots\theta_{\alpha_N}(M_N)\,dM_1\ldots dM_N \tag{3.3}$$

Thus, the problem of turbulence can be reduced to the determination of all the moments of the velocity field. These moments can be found from the dynamic equations obtained, for example, by computing the variational derivatives at zero of the right and left sides of Eq. (1.6) or (1.8) for the characteristic functional or the derivatives at zero with respect to the arguments θ_{mj} of the right and left sides of Eq. (2.8) for the characteristic function of the finite-dimensional probability distribution. Such dynamic equations can also be derived by computing directly the derivatives of the moments with respect to time with the aid of the Navier-Stokes equations. This method was used by Friedmann and Keller [9], who were the first authors to give a complete formulation of the turbulence problem (in terms of moments). The Friedmann-Keller equations for the moments $B_{j_1,\ldots,j_n}(M_1,\ldots,M_n)$ for distinct $x_1,\ldots, x_n$ and distinct $t_1,\ldots, t_n$ are

$$\left(\frac{\partial}{\partial t_m} - \nu\Delta_{\mathbf{x}_m}\right) B_{j_1,\ldots,j_n}(M_1,\ldots,M_n) = \tag{3.4}$$

$$= -\frac{\partial}{\partial x_{m\alpha}} B_{j_1,\ldots,j_{m-1}j_m\alpha j_{m+1},\ldots,j_n}(M_1,\ldots,M_{m-1},M_m,M_m,M_{m+1},\ldots,M_n) +$$

$$+ \frac{\partial}{\partial x_{mj_m}}\Delta^{-1}(\mathbf{x}_m,\mathbf{x})\frac{\partial^2}{\partial x_\alpha\partial x_\beta} B_{j_1,\ldots,j_{m-1}\alpha\beta j_{m+1},\ldots,j_n}(M_1,\ldots,M_{m-1},M,M,M_{m+1},\ldots,M_n)$$

where, as above, $M = (\mathbf{x}, t_m)$. For similar moments with equal time arguments $t_1 = \ldots = t_n = t$ the dynamic equations can be obtained by summing Eqs. (3.4) over all m with allowance for Eq. $\Sigma\partial B/\partial t_m = \partial B/\partial t$, which is valid in this case. The equations for the moments are yet another form of the equations of turbulent motion. Their analytic form is simplest, but they are always open; the equations for the n-th order moments always contain $(n+1)$-th order moments. For clarity let us write out the simplest of the Friedmann-Keller equations. In the case $n = 1$ Eqs. (3.4) become

$$\left(\frac{\partial}{\partial t}-\nu\triangle_{\mathbf{x}}\right)\langle u_j(\mathbf{x},\ t)\rangle=-\frac{\partial}{\partial x_\alpha}\langle u_j(\mathbf{x},\ t)\,u_\alpha(\mathbf{x},\ t)\rangle+ \\ +\frac{\partial}{\partial x_j}\triangle^{-1}(\mathbf{x},\ \mathbf{x}')\frac{\partial^2}{\partial x_\alpha'\partial x_\beta'}\langle u_\alpha(\mathbf{x}',\ t)\,u_\beta(\mathbf{x}',t)\rangle \tag{3.5}$$

These equations, which can be derived simply by averaging the Navier-Stokes equations were derived by Reynolds in 1894 and are often called the Reynolds equations. Using the notation $\mathbf{u}'=\mathbf{u}-\langle\mathbf{u}\rangle$ for the pulsations (deviations from the average) and applying Formula $\langle u_\alpha u_\beta\rangle=\langle u_\alpha\rangle\langle u_\beta\rangle+\langle u_\alpha' u_\beta'\rangle$, we note that in addition to the average velocity $\langle\mathbf{u}\rangle$ Eqs. (3.5) also contain the new unknowns $\langle u_\alpha' u_\beta'\rangle$ (the quantities $\tau_{\alpha\beta}=-\rho\langle u_\alpha' u_\beta'\rangle$ are called the Reynolds stresses). Further, for $n=2$ and $t_1=t_2=t$ from (3.4) we obtain

$$\left[\frac{\partial}{\partial t}-\nu(\triangle_{\mathbf{x}_1}+\triangle_{\mathbf{x}_2})\right]B_{ij}(M_1,\ M_2)= \\ =-\frac{\partial}{\partial x_{1\alpha}}B_{i\alpha j}(M_1,\ M_1,\ M_2)-\frac{\partial}{\partial x_{2\alpha}}B_{ij\alpha}(M_1,\ M_2,\ M_2)+ \\ +\frac{\partial}{\partial x_{1i}}\triangle^{-1}(\mathbf{x}_1,\ \mathbf{x})\frac{\partial^2}{\partial x_\alpha\partial x_\beta}B_{\alpha\beta j}(M,\ M,\ M_2)+\frac{\partial}{\partial x_{2j}}\triangle^{-1}(\mathbf{x}_2,\ \mathbf{x})\frac{\partial^2}{\partial x_\alpha\partial x_\beta}B_{i\alpha\beta}(M_1,\ M,\ M) \tag{3.6}$$

These equations become markedly simpler in the case of isotropic turbulence: the terms in the third line of Formula (3.6) vanish, the second and third moments in the first two lines depend only on $\mathbf{r}=\mathbf{x}_2-\mathbf{x}_1$, and the only independent equation of (3.6) is

$$\left[\frac{\partial}{\partial t}-2\nu\left(\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}\right)\right]B_{LL}(r,\ t)=\left(\frac{\partial}{\partial r}+\frac{4}{r}\right)B_{LL,\,L}(r,\ t) \tag{3.7}$$

where the subscript L corresponds to the direction along the vector r. Eq. (3.7) is called the Kármán-Howarth equation (1938). Finally, let us write out the Friedmann-Keller equations for the one-point second moments; they turn out to be more complex than (3.6), and it is convenient to write them in the form

$$\frac{\partial\langle u_i'u_j'\rangle}{\partial t}+\frac{\partial}{\partial x_\alpha}\left[\langle u_i'u_j'\rangle\langle u_\alpha\rangle+\langle u_i'u_j'u_\alpha'\rangle+\frac{1}{\rho}(\langle p'u_i'\rangle\delta_{j\alpha}+\langle p'u_j'\rangle\delta_{i\alpha})-\right. \\ \left.-(\langle u_i'\sigma_{j\alpha}'\rangle+\langle u_j'\sigma_{i\alpha}'\rangle)\right]=\langle u_i'f_j'\rangle+\langle u_j'f_i'\rangle+\left\langle\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)\right\rangle- \\ -\left(\left\langle\sigma_{i\alpha}'\frac{\partial u_j'}{\partial x_\alpha}\right\rangle+\left\langle\sigma_{j\alpha}'\frac{\partial u_i'}{\partial x_\alpha}\right\rangle\right)-\left(\langle u_i'u_\alpha'\rangle\frac{\partial\langle u_j\rangle}{\partial x_\alpha}+\langle u_j'u_\alpha'\rangle\frac{\partial\langle u_i\rangle}{\partial x_\alpha}\right) \tag{3.8}$$

Here f_i are the components of the acceleration due to external forces which we introduce for the sake of greater generality; $\sigma_{ij}=\nu(\partial u_i/\partial x_j+\partial u_j/\partial x_i)$ is the viscous stress tensor The terms containing p', f_i', and σ_{ij}', as well as the third moments $\langle u_i' u_j' u_\alpha'\rangle$ cannot be expressed directly in terms of $\langle u_i' u_j'\rangle$ and are therefore "extraneous" in Eqs. (3.8).

4. Approximate closure of the equations for moments. The simplest method of approximate closure of equations for moments involves neglecting the $(n+1)$-th order moments in the equations for the n-th order moments. The first approximation $(n=1)$ in this case would be that in which the second moments $\langle u_\alpha' u_\beta'\rangle$ (the Reynolds stresses) are neglected in the equations for the first moments (3.5). But this would mean using the ordinary Navier-Stokes equations for the average velocity field $\langle\mathbf{u}(\mathbf{x},\ t)\rangle$, i.e. neglecting turbulence completely. This is clearly unsatisfactory. The second approximation $(n=2)$ involves neglecting the third moments in the equations for the second moments (3.6). This approximation is valid for very weak turbulence only, e.g. for the final stage of degeneration of isotropic turbulence behind a wind tunnel grating. In this case the second moments are given by equations which follow from the Navier-Stokes equations if all the nonlinear terms in the latter are neglected. Since the nonlinear terms define the distribution of turbulence energy over the spectrum of the turbulence scale, it is impossible to determine the form of the turbulence spectrum in this second approximation, i.e. it must be specified in the initial conditions (corresponding to the beginning of the final stage of turbulence degeneration). In the case of isotropic turbulence this approximation can be obtained by replacing the right side of Eq. (3.7) by zero. The most general solutions of the resulting equation for $B_{LL}(r, t)$ were investigated by Sedov [10 and 11].

The second method of approximate closure consists in assuming that the $(n+1)$-th order semiinvariants vanish. This enables us to express the $(n+1)$-th order moments in terms of lower-order moments and thereby to close the Friedmann-Keller system for moments of up to order n, inclusive. (This method is related to the Kirkwood method in statistical mechanics and to the Tamm-Dankov method in quantum field theory). The first nontrivial application of the method occurs with $n = 3$, when it reduces to applying the hypothesis of Millionshchikov [12] on the equality to zero of the fourth-order semiinvariants of the velocity field; here the fourth-order moments are expressed in terms of the second-order moments by means of Formulas valid for the many-dimensional normal distribution

$$\langle w_1'w_2'w_3'w_4'\rangle = \langle w_1'w_2'\rangle\ \langle w_3'w_4'\rangle + \langle w_1'w_3'\rangle\ \langle w_2'w_4'\rangle + \langle w_1'w_4'\rangle\ \langle w_2'w_3'\rangle \tag{4.1}$$

This hypothesis involves no assumption of weak turbulence. It agrees satisfactorily with empirical data for large-scale turbulence components (but is hardly suitable for describing small-scale components).

The third method involves the use of hypotheses on the self-similarity of the statistical characteristics of turbulent flow. The most common self-similarity hypothesis (first advanced, although in slightly different form, by Kármán [13]) presumes that it is possible to introduce in the neighborhood of each point M_0 of turbulent flow length scales $l(M_0)$ and velocity scales $b(M_0)$ such that the statistical turbulence characteristics (i.e. those which are not directly affected by molecular viscosity) introduced by way of these scales are at least approximately universal (i.e. equal for all points M_0) functions of the dimensionless Galilean coordinates $\xi = [\mathbf{x} - \mathbf{x}_0 - \langle \mathbf{u}(M_0)\rangle\ (t - t_0)]/l(M_0)$. The turbulence scale l can be defined, for example, as the average length of the "mixing path" or as the "correlation radius" of the velocity field. The velocity field can represent, for example, the "turbulence intensity", i.e. the mean-square value $[\langle|\mathbf{u}'|^2\rangle]^{1/2}$ of the velocity pulsations.

Applying this hypothesis to the profile $\langle \mathbf{u}(z)\rangle$ of the average velocity of steadystate plane-parallel turbulent flow, we follow Loitsianskii [14] in requiring fulfillment of the condition

$$\frac{\langle u(z)\rangle - \langle u(z_0)\rangle}{\langle u(z_0 + l)\rangle - \langle u(z_0)\rangle} = f\left(\frac{z - z_0}{l}\right)$$

let us say to within small third-order terms in $(z - z_0)/l$. This requirement yields the Kármán formula for l

$$l = -\varkappa \frac{\partial \langle u\rangle}{\partial z} \Big/ \frac{\partial^2 \langle u\rangle}{\partial z^2}$$

Zilitinkevich and Laikhtman [15] suggest the following generalization of the Kármán formula for temperature-stratified flow:

$$l = -\frac{\varkappa\psi}{\partial\psi/\partial z}, \qquad \psi = \sqrt{\left(\frac{\partial\langle u\rangle}{\partial z}\right)^2 + \left(\frac{\partial\langle v\rangle}{\partial z}\right)^2 - \alpha\frac{g}{\langle\theta\rangle}\frac{\partial\langle\theta\rangle}{\partial z}} \tag{4.2}$$

Here g is the gravitational acceleration, $\langle\theta\rangle$ is the potential temperature, and α is a constant.

With reference to the two-point moments of isotropic turbulence appearing in the Kármán-Howarth Eqs. (3.7) the Kármán self-similarity hypothesis can be written as

$$B_{LL}(r, t) = b^2 f_1\left(\frac{r}{l}\right), \qquad B_{LL,\,L}(r, t) = b^3 f_2\left(\frac{r}{l}\right) \tag{4.3}$$

where $f_1(\xi)$ and $f_2(\xi)$ are some universal functions. Here b and l are some functions of time t. Solutions of the form (4.3) of Eq. (3.7) have been investigated most extensively by Sedov [10 and 11].

Kolmogorov [1] proposed special hypotheses for the statistical characteristics of small scale components of developed turbulence. These components exist in a state of statistical equilibrium between the forces of inertia and viscosity (the "equilibrium interval" of scales is small as compared with the scales of the flow as a whole). Specifically, Kolmogorov suggested that this state could depend on only two constant parameters, i.e. on the rate ε of turbulent energy dissipation and on the viscosity coefficient ν, so that $l=(\nu^3/\varepsilon)^{1/4}$ and $b=(\nu\varepsilon)^{1/4}$. For very large Reynolds numbers the "equilibrium interval" of the scales is so long that the scales in its large-scale portion exceed by many times the minimal scale of turbulent inhomogeneities $(\nu^3/\varepsilon)^{1/4}$. In this large-scale part of the "equilibrium interval"(called the "inertial interval") the viscosity forces are no longer essential to turbulent motions, and the state of the latter depends on the single parameter ε. This makes it possible to determine the functional form of the statistical characteristics of the turbulence components with scales from the "inertial interval". For example, at distances r from the "inertial interval" the structural tensor of the velocity field is given by

$$\langle[u_i(\mathbf{x}+\mathbf{r})-u_i(\mathbf{x})][u_j(\mathbf{x}+\mathbf{r})-u_j(\mathbf{x})]\rangle = C(\varepsilon r)^{2/3}\left(\delta_{ij}-\frac{1}{4}\frac{r_i r_j}{r^2}\right) \qquad (4.4)$$

where C is a numerical constant.

We can also mention the similarity hypotheses of Monin and Obukhov [16 and 17] for turbulence in a temperature-stratified boundary layer with constant friction stress $-\rho\langle u'w'\rangle = \rho u_*^2$ and constant heat flux $c_p\rho\langle T'w'\rangle = q$ according to which the statistical state of the turbulence components with scales ranging from the maximal (comparable with the distance to the wall) to those in the inertial interval, inclusive, depends only on the three constant parameters u_*, $q/c_p\rho$, and $g/\langle T_0\rangle$ (the latter, called the buoyancy parameter, characterizes the Archimedean accelerations $gT'/\langle T_0\rangle$, where T is the temperature). This yields $l\sim c_p\rho T_0 u_*^3/(gq)$ and $b\sim u_*$, while for the temperature we obtain the scale $T_*\sim q/(c_p\rho u_*)$. In the case of temperature homogeneity we are left with the single parameter u_*, so that, for example, we obtain $\partial\langle u\rangle/\partial z\sim u_*/z$, whence we have the familiar logarithmic law for the velocity profile in the boundary layer. In the case of thermal convection (large $q>0$) the parameter u_* ceases to be essential, so that for the potential temperature profile, for example, we obtain

$$\frac{\partial\langle\theta\rangle}{\partial z}\sim\left(\frac{q}{c\rho}\right)^{2/3}\left(\frac{g}{T_0}\right)^{-1/3} z^{-4/3} \qquad (4.5)$$

5. Semiempirical theories of turbulence. There are numerous papers in which hypotheses on the form of the "extraneous" unknowns more general than those listed in the preceding Section have been employed. Such hypotheses are usually suggested by more or less realistic physical hypotheses and yield formulas with parameters subject to empirical determination. We need mention only a few such semiempirical hypotheses.

The simplest of these have to do with the form of the Reynolds stresses $\tau_{ij}=-\rho\langle u_i'u_j'\rangle$, which are "extraneous" unknowns in Reynolds Eqs. (3.5) (and also with the form of the turbulent heat flux $q_i=c_p\rho\langle T'u_i'\rangle$ which arises in the averaging of the heat transfer equation). By analogy with the description of molecular transfer in semiempirical formulas it is generally assumed that τ_{ij} is a linear function of the tensor of the average deformation rates $\partial u_i/\partial x_j+\partial u_j/\partial x_i$ (while q_i is a linear function of the average temperature gradient $\partial\langle T\rangle/\partial x_i$). The simplest assumption of this kind, which dates back to Boussinesq (1897) is of the form $\tau=-\rho\langle u'w'\rangle=\rho K\partial\langle u\rangle/\partial z$, where K is the turbulent viscosity coefficient (similarly, according to Taylor (1915) and Schmidt (1925) $q=c_p\rho\langle T'w'\rangle=-c_p\rho\alpha K\partial\theta/\partial z$). In Prandtl's theory (1925) we set $K=l^2\partial\langle u\rangle/\partial z$, while in Taylor's theory (1932) we set $\partial\tau/\partial z=\rho K\partial^2\langle u\rangle/\partial z^2$ with the same expression for K.

The more complex hypotheses have to do with the form of the "extraneous" unknowns in Eqs. (3.8) for one-point second moments. Kolmogorov [18] suggested that K be determined by means of the turbulent energy equation obtainable from (3.8) by summation over $i=j$, which in the case of the atmospheric boundary layer, for example, reduces to

$$-\langle u'w'\rangle\frac{\partial\langle u\rangle}{\partial z}-\langle v'w'\rangle\frac{\partial\langle v\rangle}{\partial z}-\varepsilon+\frac{g}{T_0}\langle\theta'w'\rangle-\frac{\partial}{\partial z}\left\langle\left(\frac{1}{2}u_\alpha'u_\alpha'+\frac{p'}{\rho}\right)w'\right\rangle=0 \quad (5.1)$$

where we have merely omitted a single term describing the molecular diffusion of the turbulent energy. After application of Boussinesq-type hypotheses this equation becomes

$$K\left[\left(\frac{\partial\langle u\rangle}{\partial z}\right)^2+\left(\frac{\partial\langle v\rangle}{\partial z}\right)^2\right]-\varepsilon-\frac{g}{T_0}\,\alpha K\frac{\partial\langle\theta\rangle}{\partial z}+\frac{\partial}{\partial z}\,\alpha_1 K\,\frac{\partial b^2}{\partial z}=0 \qquad (5.2)$$

Further, if we set $K\sim lb$ and $\varepsilon\sim b^3/l$ in the spirit of the Kármán self-similarity hypotheses and specify somehow the turbulence scale l, then the Reynolds equations together with Eq. (5.2) for b form a closed system. Such a system was used by Monin [19] to describe a thermically homogeneous atmospheric boundary layer in which $\partial\langle\theta\rangle/\partial z=0$ for $l=\varkappa z$ (in recent years similar studies have been carried out by several authors who have used more detailed assumptions concerning the dependence of l on z (e.g. see survey [20]).

More detailed utilization of Eqs. (3.8) is the subject of the papers by Rotta [21] and Davydov [22]. Applying Eqs. (3.8) to the atmospheric boundary layer and making use of the methods of [21] and [22], Monin [23] set

$$\left\langle\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)\right\rangle=-C_1\frac{b}{l}\left(\langle u_i'u_j'\rangle-\frac{b^2}{3}\delta_{ij}\right)-C_2\frac{b^3}{l}\left(\lambda_i\lambda_j-\frac{1}{3}\delta_{ij}\right)$$

$$\nu\langle\nabla u_i'\cdot\nabla u_j'\rangle=[(C_3-3C_4)\lambda_i\lambda_j+C_4\delta_{ij}]\frac{b^3}{l} \qquad (5.3)$$

and employed similar formulas for the quantities $\langle p'_1/\rho\ \partial T'/\partial x_i\rangle$, $\langle\nabla u_i'\cdot\nabla T'\rangle$ and $\chi\langle(\nabla T')^2\rangle$ arising in Eqs. of the type (3.8) for $\langle u_i'T'\rangle$ and $\langle T'^2\rangle$ (here λ_i is the unit vector in the vertical direction). When certain small terms in the left sides of the equations are also neglected, the only "extraneous" unknown remaining in them is the turbulence scale l. Considering the latter to be given, Monin [23] succeeded in determining the velocity profiles $\langle u(z)\rangle$ and $\langle v(z)\rangle$, the temperature profile $\langle\theta(z)\rangle$, and all the second moments of the velocity and temperature pulsations (including the components of the horizontal turbulent heat flux $q_x=c_p\rho\langle T'u'\rangle$ and $q_y=c_p\rho\langle T'v'\rangle$, which are generally different from zero in a temperature-stratified medium, as has been verified by direct measurements of these quantities.

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